Chapter 21 – Discontinuities and Limits

Some functions are defined for all values of \( x \). This means their graphs go on forever with no breaks. These are called continuous functions.

Examples:
\[
\begin{align*}
  f(x) &= 2x + 1 \\
  g(x) &= x^2 - 3
\end{align*}
\]

Other functions have some \( x \)-values that are excluded from their domains. Such functions will have breaks in their graphs at the \( x \)-values that don’t work. These functions are called discontinuous.

Examples:
\[
\begin{align*}
  f(x) &= \frac{2x+1}{x-3} \\
  g(x) &= \frac{x^2 - 5x + 6}{x-3}
\end{align*}
\]

Sometimes we can determine what the \( y \)-value of a function should be at a particular \( x \)-value, even if the function is discontinuous at that \( x \)-value. This expected \( y \)-value is called the limit. (Note that the limit might actually be the \( y \)-value or it might not!)
Limits can be calculated algebraically.

Example: Evaluate each limit algebraically:

a) \( \lim_{x \to 2} (3x + 7) = 3 \cdot 2 + 7 = 13 \)

b) \( \lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x \to 4} \frac{(x + 3)(x - 4)}{x - 4} = \lim_{x \to 4} (x + 3) = 4 + 3 = 7 \)

c) \( \lim_{x \to 0} \frac{x^3 - 5x^2 + 9x}{x} = \lim_{x \to 0} \frac{x(x^2 - 5x + 9)}{x} = \lim_{x \to 0} (x^2 - 5x + 9) = 9 \)

Some functions have a discontinuity, but the function does not approach a particular \( y \)-value at that point. In this case we say that the limit does not exist.
Chapter 21 – Introduction to the Derivative

- You already know that for a linear function the rate of change is constant, and this rate is represented by the slope of the line:

\[
\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

- For a non-linear function, the rate of change varies at different x-values along the curve. The rate of change of a non-linear function at a particular point is represented by the slope of the tangent line to the curve at that point.

- Since the slope of the tangent line changes at different x-values, it is a function of x. The slope function is called the derivative.

- The derivative of a function \( f(x) \) is denoted \( f'(x) \) and it can be calculated using the following definition:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

- Finding the derivative of a function from this definition is referred to as using first principles. Later you will learn easier methods for finding derivatives.

Example: Find the derivative of \( f(x) = x^2 \) using first principles.

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h} = 2x
\]

Example: Find the derivative of \( f(x) = x^3 - 2x + 3 \) using first principles.

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - 2(x + h) + 3 - (x^3 - 2x + 3)}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h + 3 - x^3 + 2x - 3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 2)}{h} = 3x^2 - 2
\]
Chapter 21 – Simple Differentiation Rules

➢ To differentiate a function means to find its derivative.

➢ When the original function is written using \( f(x) \) notation, the derivative is denoted \( f'(x) \), pronounced “f prime of \( x \).”

➢ When the original function is written using \( y \), the derivative is denoted \( \frac{dy}{dx} \), pronounced “d-y-d-x.”

➢ Finding derivatives of simple functions, like polynomials, is easy. Some rules for finding simple derivatives are summarized in the box below:

<table>
<thead>
<tr>
<th>Name of Rule</th>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative of a power of ( x )</td>
<td>( f(x) = x^n )</td>
<td>( f'(x) = nx^{n-1} )</td>
</tr>
<tr>
<td>Derivative of a constant</td>
<td>( f(x) = k )</td>
<td>( f'(x) = 0 )</td>
</tr>
<tr>
<td>Derivative of a constant times a power of ( x )</td>
<td>( f(x) = kx^n )</td>
<td>( f'(x) = knx^{n-1} )</td>
</tr>
<tr>
<td>Derivative of a sum or difference of functions</td>
<td>( p(x) = f(x) \pm g(x) )</td>
<td>( p'(x) = f'(x) \pm g'(x) )</td>
</tr>
</tbody>
</table>

Example: Find the derivative of \( f(x) = 4x^3 - 7x^2 + 2x - 8 \).

\[
f'(x) = 12x^2 - 14x + 2
\]

Example: Find the derivative of \( f(x) = \frac{1}{x^2} + 6\sqrt{x} - 5 \).

\[
f'(x) = -2x^{-3} + 3x^{-\frac{1}{2}}
\]
Chapter 21 – Tangent Lines

➢ For a function \( f(x) \), the derivative \( f'(x) \) gives the slope of the tangent at a particular point. The point on the curve \( f(x) \) where the tangent is drawn is called the point of tangency.

➢ To find the equation of the tangent line to \( f(x) \) at a given \( x \)-value, follow these steps:

1) Find the \( y \)-coordinate of the point of tangency by plugging \( x \) into the function \( f(x) \).
2) Find the derivative \( f'(x) \).
3) Find the slope of the tangent \( (m) \) by plugging \( x \) into the derivative \( f'(x) \).
4) Find the \( y \)-intercept \( (b) \) of the tangent line by plugging \( x, y, \) and \( m \) into the general equation of a line, \( y = mx + b \).
5) Write the equation in \( y = mx + b \) form.

Example: Find the equation of the tangent to \( f(x) = 3x^2 - 5x + 3 \) at the point where \( x = 2 \).

\[
\begin{align*}
y &= f(2) = 3(2)^2 - 5(2) + 3 = 5 \\
f'(x) &= 6x - 5 \quad \rightarrow \quad m = f'(2) = 6(2) - 5 = 7 \\
y &= mx + b \quad \rightarrow \quad 5 = 2(7) + b \\
      &= 14 + b \\
      &= 14 + (-9) \\
      &= 5 \\
b &= -9
\end{align*}
\]

The equation of the tangent line is \( y = 7x - 9 \).
Chapter 21 – Derivatives and Tangent Lines on the GDC

- You can use your graphing calculator to find the derivative of a function at a given \( x \)-value and to find the equation of the tangent line at that point.

- The first step for both is to graph the function and set your viewing window so that the desired \( x \)-value is visible on the graph. Then…

…to find the derivative:
- From the graph window, go to the calculate menu (2nd – TRACE)
- Choose 6: dy/dx
- Type in the \( x \)-value
- The calculator will display the derivative at that point. (Remember sometimes the calculator has some rounding error!)

…to find the equation of the tangent:
- From the graph window, go to the draw menu (2nd – PRGM)
- Choose 5: Tangent(
- Type in the \( x \)-value
- The calculator will graph the tangent line at that point and display its equation. (Remember sometimes the calculator has some rounding error!)

Example: Use your GDC to the derivative of \( f(x) = 3\ln(x+5) \) at the point where \( x = -3 \).

The derivative at \( x = -3 \) is \( f'(-3) = 1.5 \).

Example: Use your GDC to find the equation of the tangent line to \( g(x) = \sqrt{3x-1} \) at the point where \( x = 1 \). (Round your answer to three decimal places.)
The equation of the tangent line is \( y = 1.061x + 0.354 \).
Chapter 21 – The Chain Rule

➢ Recall that two functions \( f \) and \( g \) can be combined by finding the composite function \( g(f(x)) \). Here \( f \) is called the inside function and \( g \) is called the outside function.

➢ The chain rule is used to find the derivative of a composite function. Let \( y = g(u) \) where \( u = f(x) \). Then \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \).

Example: Find the derivative of \( y = (1 - 3x)^4 \).

\[
\begin{align*}
y &= u^4 & u &= 1 - 3x \\
\frac{dy}{du} &= 4u^3 & \frac{du}{dx} &= -3 \\
\frac{dy}{dx} &= 4u^3(-3) = -12u^3 = -12(1 - 3x)^3
\end{align*}
\]

Example: Find the derivative of \( y = \sqrt{x^2 - 5x} \).

\[
\begin{align*}
y &= u^{\frac{1}{2}} & u &= x^2 - 5x \\
\frac{dy}{du} &= \frac{1}{2}u^{-\frac{1}{2}} & \frac{du}{dx} &= 2x - 5 \\
\frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}}(2x - 5) = \frac{1}{2}(x^2 - 5x)^{-\frac{1}{2}}(2x - 5) &= \frac{2x - 5}{2\sqrt{x^2 - 5x}}
\end{align*}
\]
Chapter 21 – Product and Quotient Rules

- The **product rule** is used to find the derivative of a function formed by multiplying two functions together. Let \( y = uv \) where \( u \) and \( v \) are both functions of \( x \). Then \( \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \).

Example: Differentiate \( y = (x^2 + x + 1)(x^2 + 2) \).

\[
\begin{align*}
  u &= x^2 + x + 1, & v &= x^2 + 2 \\
  \frac{du}{dx} &= 2x + 1, & \frac{dv}{dx} &= 2x \\
  \frac{dy}{dx} &= (x^2 + x + 1)(2x) + (x^2 + 2)(2x + 1)
\end{align*}
\]

- The **quotient rule** is used to find the derivative of a function formed by dividing two functions. Let \( y = \frac{u}{v} \) where \( u \) and \( v \) are both functions of \( x \). Then \( \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \).

Example: Find the derivative of \( y = \frac{\sqrt{x}}{1 - 2x} \).

\[
\begin{align*}
  u &= x^{\frac{1}{2}}, & v &= 1 - 2x \\
  \frac{du}{dx} &= \frac{1}{2} x^{-\frac{1}{2}}, & \frac{dv}{dx} &= -2 \\
  \frac{dy}{dx} &= \frac{(1 - 2x)\left(\frac{1}{2} x^{-\frac{1}{2}}\right) - \left(\sqrt{x}\right)(-2)}{(1 - 2x)^2}
\end{align*}
\]
Chapter 21 – Normal Lines

- A normal line is a line perpendicular to the tangent line at the point of tangency.

- Recall that the slopes of perpendicular lines are opposite reciprocals. For example, the lines $y = \frac{1}{3}x + 7$ and $y = -3x + 9$ are perpendicular.

- Finding the equation of a normal line is very similar to finding the equation of the tangent line, just follow these steps:
  1) Find the $y$-coordinate of the point of tangency by plugging $x$ into the function $f(x)$.
  2) Find the derivative $f'(x)$.
  3) Find the slope of the tangent ($m_t$) by plugging $x$ into the derivative $f'(x)$.
  4) Find the slope of the normal ($m_n$) by taking the opposite reciprocal of the slope of the tangent.
  5) Find the $y$-intercept ($b$) of the normal line by plugging $x$, $y$, and $m_n$ into the general equation of a line, $y = mx + b$.
  6) Write the equation in $y = mx + b$ form.

Example: Find the equation of the normal to $f(x) = 3x^2 - 5x + 3$ at the point where $x = 2$.

$$y = f(2) = 3(2)^2 - 5(2) + 3 = 5$$
$$f'(x) = 6x - 5 \quad \rightarrow \quad m_t = 7 \quad \rightarrow \quad m_n = -\frac{1}{7}$$
$$y = mx + b \quad \rightarrow \quad 5 = 2\left(-\frac{1}{7}\right) + b$$
$$5 = -\frac{2}{7} + b \quad \rightarrow \quad b = \frac{37}{7}$$

The equation of the normal line is $y = -\frac{1}{7}x + \frac{37}{7}$. 
Chapter 21 – The second Derivative

- The second derivative is simply the derivative of the derivative. So to find the second derivative of a function, just differentiate twice!

- The second derivative is denoted $f''(x)$, pronounced “f double-prime of x” or $\frac{d^2y}{dx^2}$, pronounced “d two y d x-squared.”

Example: Find the second derivative of $f(x) = 5x^4 - 3x^2 + 7x$

$$f'(x) = 20x^3 - 6x$$
$$f''(x) = 60x^2 - 6$$

Example: Find the second derivative of $y = 8\sqrt{x^3}$

$$y = 8x^{3/2}$$
$$\frac{dy}{dx} = 8 \cdot \frac{3}{2} x^{-1/2} = 12x^{-1/2}$$
$$\frac{d^2y}{dx^2} = \frac{12}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} = -6x^{-3/2}$$

- We will look at applications of the second derivative in the next chapter.
Chapter 22 – Motion in a Straight Line

- Suppose an object moves in a straight line from a starting point $O$, and that its position $s$ is given as a function of time, $t$. The function $s(t)$ is called the displacement function for the object.

- The sign of the function $s$ indicates the position of the object relative to the origin point:
  - If $s(t) = 0$ the object is at the origin point.
  - If $s(t) > 0$ the object is to the right of the origin point.
  - If $s(t) < 0$ the object is to the left of the origin point.

- The average velocity of the object between $t_1$ and $t_2$ is $\frac{s(t_1) - s(t_2)}{t_1 - t_2}$.

- The instantaneous velocity, or velocity function, is $v(t) = s'(t)$.

- The instantaneous acceleration, or acceleration function, is $a(t) = v'(t) = s''(t)$.

- $s(0)$, $v(0)$, and $a(0)$ give us the position, velocity, and acceleration of the object at time $t = 0$. These are called the initial conditions.

- You will often encounter the following phrases in motion problems, and you need to know how to interpret them:
  - “The object is at the origin point” means $s(t) = 0$.
  - “The object is stationary” means $v(t) = 0$.
  - “The object reverses direction” means $v(t) = 0$.
  - “The object reaches its maximum/minimum height” means $v(t) = 0$.
  - “The velocity is constant” means $a(t) = 0$.
  - “The object reaches its maximum/minimum velocity” means $a(t) = 0$. 
Chapter 22 – Curve Properties

- For a function \( f(x) \), the derivative can be used to help determine the shape of the graph:
  
  - If \( f'(x) > 0 \) then \( f(x) \) is increasing.
  - If \( f'(x) < 0 \) then \( f(x) \) is decreasing.

- If \( f'(a) = 0 \) then the point where \( x = a \) is called a stationary point. A stationary point can be a local maximum, a local minimum, or a horizontal inflection point.

- A sign diagram for \( f'(x) \) can be used to determine the intervals of increase/decrease and to find and classify stationary points. The chart below shows how to interpret a sign diagram for \( f'(x) \):

<table>
<thead>
<tr>
<th>Sign Diagram</th>
<th>Type of Point</th>
<th>Graph</th>
</tr>
</thead>
</table>
| \[ \begin{array}{c}
\nearrow & + & \searrow \\
\nearrow & a & \searrow
\end{array} \] \( f'(x) \) | Local maximum at \( x = a \). | ![Graph of Local Maximum] |
| \[ \begin{array}{c}
\nearrow & - & \searrow \\
\nearrow & a & \searrow
\end{array} \] \( f'(x) \) | Local minimum at \( x = a \). | ![Graph of Local Minimum] |
| \[ \begin{array}{c}
\nearrow & + & \searrow \\
\nearrow & a & \searrow
\end{array} \] \( f'(x) \) or \[ \begin{array}{c}
\nearrow & - & \searrow \\
\nearrow & a & \searrow
\end{array} \] \( f'(x) \) | Horizontal inflection at \( x = a \). | ![Graph of Horizontal Inflection] |
Example: Find and classify all stationary points on the graph of 
\( f(x) = 3x^4 - 8x^3 + 2 \), and state the intervals of 
increase/decrease.

\[
\begin{align*}
  f'(x) &= 12x^3 - 24x^2 \\
  12x^3 - 24x^2 &= 0 \\
  12x^2(x - 2) &= 0 \\
  x &= 0 \quad \text{and} \quad x = 2
\end{align*}
\]

At (0, 2) \( f(x) \) has a horizontal inflection point.
At (2, -14) \( f(x) \) has a local maximum.
\( f(x) \) is decreasing when \( x < 2 \) and increasing when \( x > 2 \).

Example: Find and classify all stationary points on the graph of 
\( f(x) = x^3 - 3x^2 - 9x + 5 \), and state the intervals of 
increase/decrease.

\[
\begin{align*}
  f'(x) &= 3x^2 - 6x - 9 \\
  3(x^2 - 2x - 3) &= 0 \\
  3(x - 3)(x + 1) &= 0 \\
  x &= 3 \quad \text{and} \quad x = -1
\end{align*}
\]

At (-1, 10) \( f(x) \) has a local maximum.
At (3, -22) \( f(x) \) has a local minimum.
\( f(x) \) is increasing when \( x < -1 \), decreasing when \( -1 < x < 3 \),
and increasing when \( x > 3 \).
Chapter 22 – Inflections and Shape Type

- For a function \( f(x) \), the second derivative can be used to determine the concavity of the graph:
  - If \( f''(x) > 0 \) then \( f(x) \) is concave up.
  - If \( f''(x) < 0 \) then \( f(x) \) is concave down.

- An easy way to remember these facts:

\[
\_\_\_\_ + + \_\_\_\_ - -
\]

- If \( f''(a) = 0 \) and the sign of \( f''(x) \) changes at \( x = a \), then the graph of \( f(x) \) has an inflection point at \( x = a \). However, to determine the type of inflection you must look at the sign of the first derivative:
  - If \( f'(a) = 0 \) then it is a horizontal inflection point.
  - If \( f'(a) \neq 0 \) then it is a non-horizontal inflection point.

- A sign diagram for \( f''(x) \) can be used to determine the concavity of \( f(x) \) and to find and classify inflection points.

Example: Find and classify all points of inflection on the graph of \( f(x) = x^4 - 4x^3 + 5 \), and state the intervals of concavity.

\[
f'(x) = 4x^3 - 12x^2 \quad \text{and} \quad f''(x) = 12x^2 - 24x
\]

\[
12x^2 - 24x = 0
\]

\[
12x(x - 2) = 0
\]

\[
x = 0 \quad \text{and} \quad x = 2
\]

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
0 & 2 & t''(x)
\end{array}
\]

\[
f'(0) = 0 \quad \text{and} \quad f'(2) = -16 \neq 0
\]

At (0, 5) \( f(x) \) has a horizontal inflection point.
At (2, -11) \( f(x) \) has a non-horizontal inflection point.
\( f(x) \) is concave up when \( x < 0 \), concave down when \( 0 < x < 2 \), and concave up when \( x > 2 \).
Chapter 22 – Rational Functions
Recall that an exponential function has the form \( y = ab^x \). In calculus, the exponential function with base \( e \) is of particular interest.

The number \( e = 2.718281828459045\ldots \) is an irrational constant. It is also the base of the natural logarithm function \( y = \ln x \).

The function \( y = e^x \) is very important in calculus because it has a unique property – it is its own derivative, that is, \( \frac{dy}{dx} = e^x \).

To differentiate a function of the form \( y = e^{f(x)} \), apply the chain rule:

\[
\frac{dy}{dx} = f'(x) \cdot e^{u(x)}
\]

Examples: Differentiate the following functions:

1) \( y = e^{5x} \)
2) \( y = e^{x^2-2x} \)
3) \( y = 2e^x - e^{-3x^2} \)

1) \( y = e^u \) \( u = 5x \)
\[
\frac{dy}{du} = e^u \quad \frac{du}{dx} = 5
\]
\[
\frac{dy}{dx} = 5 \cdot e^u = 5e^{5x}
\]

2) \( y = e^u \) \( u = x^2 - 2x \)
\[
\frac{dy}{du} = e^u \quad \frac{du}{dx} = 2x - 2
\]
\[
\frac{dy}{dx} = (2x - 2) \cdot e^u = (2x - 2)e^{x^2-2x}
\]

3) \( y = 2e^x - e^{-3x^2} \)
\[
\frac{dy}{dx} = 2e^x + 6e^{-3x^2}
\]
Chapter 23 – Derivatives of Logarithmic Functions

➢ The logarithmic function with base $e$ is $y = \ln x$. The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$.

➢ To differentiate a function of the form $y = \ln(f(x))$, apply the chain rule:

$$y = \ln u \quad u = f(x)$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = f'(x)$$

$$\frac{dy}{dx} = f'(x) \cdot \frac{1}{u} = f'(x) \cdot \frac{1}{f(x)} = \frac{f'(x)}{f(x)}$$

Example: Differentiate $y = \ln(x^2 + 2x)$

$$y = \ln u \quad u = x^2 + 2x$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x}$$

➢ Sometimes it is helpful to use properties of logarithms to expand a function first before you find the derivative.

Example: Differentiate $y = \ln\left(x(x^2 + 1)\right)$

$$y = \ln\left(x(x^2 + 1)\right) = \ln x + \ln(x^2 + 1)$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{x + 1}$$

Example: Differentiate $y = \ln\left(\frac{x^3}{(2-3x)^2}\right)$

$$y = \ln\left(\frac{x^3}{(2-3x)^2}\right) = \ln(x^3) - \ln(2-3x)^2 = 3\ln x - 2\ln(2-3x)$$

$$\frac{dy}{dx} = 3 \cdot \frac{1}{x} - 2 \cdot \frac{-3}{(2-3x)} = \frac{3}{x} + \frac{6}{2-3x}$$
The derivatives of trigonometric functions where \( x \) is measured in radians are as follows:

\[
\begin{align*}
  y &= \sin x \quad \frac{dy}{dx} = \cos x \\
  y &= \cos x \quad \frac{dy}{dx} = -\sin x \\
  y &= \tan x \quad \frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x
\end{align*}
\]

The derivative of \( y = \sin (f(x)) \) can be found using the chain rule:

\[
\frac{dy}{dx} = f'(x) \cdot \cos u = f'(x) \cos (f(x))
\]

Similarly, the derivative of \( y = \cos (f(x)) \) is \( \frac{dy}{dx} = -f'(x) \sin (f(x)) \) and the derivative of \( y = \tan (f(x)) \) is \( \frac{dy}{dx} = \frac{f'(x)}{\cos^2 (f(x))} \).

Examples: Differentiate the following functions:

1) \( y = \sin x^2 \) 
2) \( y = \tan 2x \) 
3) \( y = \sin^2 x \)

1) \( \frac{dy}{dx} = 2x \cos x^2 \)

2) \( \frac{dy}{dx} = \frac{2}{\cos^2 2x} \)

3) \( \frac{dy}{dx} = 2 \sin x \cos x \) (using the chain rule)